OPTIMAL CONTROL OF A LARGE DAM

VYACHESLAV M. ABRAMOV

Abstract. A large dam model is an object of study of this paper. The parameters L^{lower} and L^{upper} are its lower and upper levels, $L = L^{upper} - L^{lower}$ is large, and if a current level of water is between these bounds, then the dam is assumed to be in normal state. Passage one or other bound leads to damage. Let J_1 (J_2) denote the damage cost of crossing the lower (upper) level. It is assumed that input stream of water is described by a Poisson process, while the output stream is state-dependent (the exact formulation of the problem is given in the paper). Let L_t denote the dam level at time t, and let $p_1 = \lim_{t\to\infty} \mathbf{P}\{L_t = L^{lower}\}$, $p_2 = \lim_{t\to\infty} \mathbf{P}\{L_t > L^{upper}\}$ exist. The long-run average cost $J = p_1J_1 + p_2J_2$ is a performance measure. The aim of the paper is to choose the parameter of output stream (exactly specified in the paper) minimizing J.

1. **Introduction**

A large dam model is an object of study of this paper. The parameters L^{lower} and L^{upper} are lower and upper levels of the dam, and if a current level of water is between these bounds, then the dam is assumed to be in normal state. The reason that a dam is called *large* is that the difference $L = L^{upper} - L^{lower}$ is large. This assumption enables us to use asymptotic analysis as $L \to \infty$ and obtain much simpler representations for the desired characteristics of the model. In turn these representations are then easily used to solve the appropriate control problems formulated below.

In the literature, the dam, storage and production models are associated with state-dependent queueing systems (e.g. [1], [2], [7], [8], [9], [10], [11], [12], [13], [21] and others). The model of the present paper is the following. We assume that units of water, arriving to a dam, are registered by counter at random instants t_1, t_2,..., and interarrival times $\tau_n = t_{n+1} - t_n$ are mutually independent exponentially distributed random variables with parameter λ. Outflow of water is state-dependent as follows. If the level of water is between L^{lower} and L^{upper}, then an interval between unit departures has the probability distribution $B_1(x)$. If the level of water increases above the level L^{upper}, then the probability distribution of an interval between unit departures is $B_2(x)$. It is also assumed that if the level of water is exactly

1991 *Mathematics Subject Classification.* 60K30, 40E05, 90B05, 60K25.

Key words and phrases. Dam, State-dependent queue, Asymptotic analysis, Control problem.

of the level L^{lower}, then the departure process of water is frozen, and it is resumed again as soon as the level of water increases the value L^{lower}. It is worth noting that the policies at which the service rate is changed in dependence of a dam level are of notable attention in the literature (see [1], [2], [7], [8], [21] and other papers). However all of them discuss performance measures associated with an appropriate upper level of water in a dam and, to our best knowledge, the known results on performance analysis of river flows structured by lower and upper levels are analytically difficult and hard for real applications even for simple models (e.g. see review paper [14]). Furthermore, in the most studies the explicit representations are in terms of the Laplace-Stieltjes transforms of the initial distributions, and there is no paper providing asymptotic analysis of large dams.

In terms of queueing theory the problem considered in the paper can be reformulated as follows. Consider single-server queueing system where arrival flow of customers is Poisson with rate λ, and a service time of a customer depends upon queue-length as follows. If at the moment of service begun, the number of customers in the system is not greater than L, then the service time of this customer has the probability distribution $B_1(x)$. Otherwise, if there are more than L customers in the system at the moment of service begun, then the probability distribution function of the service time of this customer is $B_2(x)$. The analytical results for this queueing system are known (e.g. Abramov [3]). Notice that the lower level of dam L^{lower} is equated with an empty queueing system. Then the dam specification of the problem is characterized by performance criteria, which in terms of the queueing formulation looks as follows. Let q_t denote the queue-length in time t. The problem is to choose an output parameter of system minimizing the functional $J(L) = p_1(L)J_1(L) + p_2(L)J_2(L)$, where $p_1(L) = \lim_{t\to\infty} \mathbf{P}\{q_t = 0\}$, $p_2(L) = \lim_{t\to\infty} \mathbf{P}\{q_t > L\}$, and $J_1(L)$ and $J_2(L)$ are the corresponding damage costs proportional to L. To be precise assume that $J_1(L) = j_1L$ and $J_2(L) = j_2L$, where j_1 and j_2 are positive constants. Assuming that $L \to \infty$ we shall often write p_1 and p_2 (without argument L) rather than $p_1(L)$ and $p_2(L)$. The argument L will be often omitted in other functions. We shall feel free to write J, J_1 and J_2 without the argument L.

To specify the problem more correctly we assume that the input parameter λ, and probability distribution function $B_2(x)$ are given, while $B_1(x) = B_1(x, C)$ is a family of probability distributions depending on parameter $C \geq 0$, which in turn is closely related with the expectation $\int_0^\infty x\mathrm{d}B_1(x)$. Then the output rate, associated with this probability distribution $B_1(x)$, can be changed so that the minimum value of the functional is associated with the choice of this parameter C, resulting in the choice of the function $B_1(x, C)$. The correctness of such formulation and more concrete clarification of parameter C will be explained in the sequel (see the formulations of Theorem 4.1, 4.2, 4.3, 4.4). It is interesting to note, *that the solution of the above control problem is asymptotically independent of the explicit form of probability distribution functions $B_1(x)$ and $B_2(x)$, and*

only depends on the expectations $\int_0^\infty x\mathrm{d}B_2(x) < \frac{1}{\lambda}$ *and* $\int_0^\infty x\mathrm{d}B_1(x)$ *as well as* $\int_0^\infty x^2\mathrm{d}B_1(x)$. The details of this dependence will be explained later.

We use the notation $b_i = \int_0^\infty x\mathrm{d}B_i(x)$, $\rho_i = \lambda b_i$ $(i = 1,2)$ and assume that $\rho_2 < 1$. This assumption is a standard condition of stationarity, ergodicity of the queue-length process q_t and existence of the limits p_1 and p_2 (independent of any initial state of the process). In additional to this assumption we shall also assume the existence of the third moment: $\rho_{1,k} = \lambda^k \int_0^\infty x^k\mathrm{d}B_1(x) < \infty$, $k = 2,3$. The existence of the second moment is used in Theorem 3.2. Then the existence of the third moment for all the specified family of distributions $B_1(x,C)$ is required in Theorem 4.1, 4.2, 4.3, 4.4.

The special features of the present paper are as follows.

• We solve the control problem where the performance criteria takes into account passage of the upper and lower bounds. The formulation of the problem is not traditional but realistic. Other similar control problems arising in practice can be flexibly solved by adapting the method of this paper.

• The presentation of the results are clear and available for real application.

• The mathematical methods of the paper are not traditional but clear and easily understandable.

The paper is structured as follows. In Section 2 we discuss the state-dependent queue-length process and derive representation for the probabilities p_1 and p_2. Section 3 contains the results on asymptotic analysis of probabilities p_1 and p_2, and the main result of this asymptotic behaviour is given by Theorem 3.2. In Section 4 some additional theorems on asymptotic behaviour of p_1 and p_2 are proved, which are then used to solve the control problem. The main result of this paper, the solution of control problem is formulated in Section 5. Concluding remark are given in Section 6.

2. **The state-dependent queue and its characteristics in a busy period**

In this section we discuss the main characteristics of the state-dependent queueing system described in the introduction. Let T_L, I_L and ν_L denote correspondingly a busy period, an idle period and the number of served customers during a busy period. Let $T_L^{(1)}$, $T_L^{(2)}$ denote the total time during a busy period when correspondingly $0 < q_t \leq L$ and $q_t > L$, and let $\nu_L^{(1)}$, $\nu_L^{(2)}$ denote correspondingly the total numbers of served customers during a busy period when correspondingly $0 < q_t \leq L$ and $q_t > L$. We have the following two obvious equations:

$$\mathbf{E}T_L = \mathbf{E}T_L^{(1)} + \mathbf{E}T_L^{(2)}, \tag{2.1}$$

$$\mathbf{E}\nu_L = \mathbf{E}\nu_L^{(1)} + \mathbf{E}\nu_L^{(2)}. \tag{2.2}$$

According to the Wald's equation,

$$\mathbf{E}T_L^{(1)} = b_1\mathbf{E}\nu_L^{(1)}, \tag{2.3}$$

and

$$\mathbf{E}T_L^{(2)} = b_2\mathbf{E}\nu_L^{(2)}. \tag{2.4}$$

Next, the number of arrivals during a busy circle coincides with the total number of served customers during a busy period. Hence, applying the Wald's equation again and taking into account (2.1)-(2.4), we obtain

$$\begin{aligned}\lambda\mathbf{E}T_L + \lambda\mathbf{E}I &= \lambda\mathbf{E}T_L + 1\\ &= \lambda\mathbf{E}T_L^{(1)} + \lambda\mathbf{E}T_L^{(2)} + 1\\ &= \rho_1\mathbf{E}\nu_L^{(1)} + \rho_2\mathbf{E}\nu_L^{(2)} + 1\\ &= \mathbf{E}\nu_L^{(1)} + \mathbf{E}\nu_L^{(2)}.\end{aligned} \tag{2.5}$$

From (2.5) we have the equation

$$\mathbf{E}\nu_L^{(2)} = \frac{1}{1-\rho_2} - \frac{1-\rho_1}{1-\rho_2}\mathbf{E}\nu_L^{(1)}, \tag{2.6}$$

expressing $\mathbf{E}\nu_L^{(2)}$ via $\mathbf{E}\nu_L^{(1)}$. For example, if $\rho_1 = 1$, then $\mathbf{E}\nu_L^{(2)} = \frac{1}{1-\rho_2}$ for any L.

The similar equation holds also for $\mathbf{E}T_L^{(2)}$. Namely, from (2.4) and (2.6) we obtain

$$\mathbf{E}T_L^{(2)} = \frac{\rho_2}{\lambda(1-\rho_2)} - \frac{\rho_2(1-\rho_1)}{\lambda(1-\rho_2)}\mathbf{E}T_L^{(1)}. \tag{2.7}$$

Equations (2.6) and (2.7) enables us to obtain the stationary probabilities p_1 and p_2. Applying the renewal reward theorem (e.g. Ross [17], p. 78) and consequently (2.5) and (2.6), for p_1 we obtain:

$$\begin{aligned}p_1 &= \frac{\mathbf{E}I}{\mathbf{E}T_L^{(1)} + \mathbf{E}T_L^{(2)} + \mathbf{E}I}\\ &= \frac{1}{\mathbf{E}\nu_L^{(1)} + \mathbf{E}\nu_L^{(2)}}\\ &= \frac{1-\rho_2}{1 + (\rho_1-\rho_2)\mathbf{E}\nu_L^{(1)}}.\end{aligned} \tag{2.8}$$

Analogously,

$$
\begin{aligned}
p_2 &= \frac{\mathbf{E}T_L^{(2)}}{\mathbf{E}T_L^{(1)} + \mathbf{E}T_L^{(2)} + \mathbf{E}I} \\
&= \frac{\rho_2 \mathbf{E}\nu_L^{(2)}}{\mathbf{E}\nu_L^{(1)} + \mathbf{E}\nu_L^{(2)}} \qquad (2.9) \\
&= \frac{\rho_2 + \rho_2(\rho_1 - 1)\mathbf{E}\nu_L^{(1)}}{1 + (\rho_1 - \rho_2)\mathbf{E}\nu_L^{(1)}}.
\end{aligned}
$$

3. Asymptotic analysis of p_1 and p_2 as L increases to infinity

By sample path analysis and the property of the lack of memory of exponential distribution it follows that the random variable $\nu_L^{(1)}$ coincides in distribution with the number of served customers during a busy period of the $M/GI/1/L$ queueing system (the parameter L denotes the number of customers in the system excluding the customer in the server). Specifically, we use the fact that during a busy period the number of times service begun when the number of customers in the system does not exceed L, coincides with the number of arrivals when the number of customers in the system does not exceed $L+1$. We also use the fact that the residual interarrival time after a service completion has exponential distribution with parameter λ.

Therefore the known results of the $M/GI/1/L$ queueing system can be used.

It is known (e.g. [3], [4]) that $\mathbf{E}\nu_L^{(1)}$ is determined by the convolution type recurrence relation

$$\mathbf{E}\nu_L^{(1)} = \sum_{j=0}^{L} \mathbf{E}\nu_{L-j+1}^{(1)} \int_0^\infty \mathrm{e}^{-\lambda x} \frac{(\lambda x)^j}{j!} \mathrm{d}B_1(x), \quad \mathbf{E}\nu_0^{(1)} = 1,$$

where $\mathbf{E}\nu_n^{(1)}$ denotes the expectation of the number of served customers during a busy period of $M/GI/1/n$ queue ($n = 1, 2, \ldots$).

The probabilities $p_1(L)$ and $p_2(L)$ are expressed explicitly via $\mathbf{E}\nu_L^{(1)}$, and their asymptotic behavior as $L \to \infty$ can be obtained from the following known results.

Let $Q_0 > 0$ be an arbitrary real number, and for $n \geq 0$

$$Q_n = \sum_{j=0}^{n} r_j Q_{n-j+1},$$

where $r_0 > 0$, $r_j \geq 0$, and $r_0 + r_1 + \ldots = 1$. Let $r(z) = \sum_{j=0}^\infty r_j z^j$, $|z| \leq 1$ be a generating function, and let $\gamma_m = \lim_{z \uparrow 1} r^{(m)}(z)$, where $r^{(m)}(z)$ is the mth derivative of $r(z)$.

Notice, that the sequence $\{Q_n\}$ is an increasing sequence, and

$$\sum_{n=0}^{\infty} Q_n z^n = \frac{Q_0 r(z)}{z - r(z)} \tag{3.1}$$

(see [15], [16], Sect. 25 and [20]).

Lemma 3.1. *(Takács [20], p. 22-23). If $\gamma_1 < 1$, then*

$$\lim_{n\to\infty} Q_n = \frac{Q_0}{1-\gamma_1}.$$

If $\gamma_1 = 1$ and $\gamma_2 < \infty$, then

$$\lim_{n\to\infty} \frac{Q_n}{n} = \frac{2Q_0}{\gamma_2}.$$

If $\gamma_1 > 1$, then

$$\lim_{n\to\infty} \left[Q_n - \frac{Q_0}{\sigma^n(1 - r'(\sigma))}\right] = \frac{1}{1-\gamma_1},$$

where σ is the least in absolute value root of functional equation $z = r(z)$.

From this lemma we have the asymptotic results for the probabilities p_1 and p_2.

For $\Re(s) \geq 0$ denote by $\widehat{B}_1(s)$ the Laplace-Stieltjes transform of $B_1(x)$. We have the following theorem.

Theorem 3.2. *If $\rho_1 < 1$, then*

$$\lim_{L\to\infty} p_1(L) = 1 - \rho_1, \tag{3.2}$$

$$\lim_{L\to\infty} p_2(L) = 0. \tag{3.3}$$

If $\rho_1 = 1$, then

$$\lim_{L\to\infty} Lp_1(L) = \frac{\rho_{1,2}}{2}, \tag{3.4}$$

$$\lim_{L\to\infty} Lp_2(L) = \frac{\rho_2}{1-\rho_2} \cdot \frac{\rho_{1,2}}{2}. \tag{3.5}$$

If $\rho_1 > 1$, then

$$\lim_{L\to\infty} \frac{p_1(L)}{\varphi^L} = \frac{(1-\rho_2)[1 + \lambda \widehat{B}_1'(\lambda - \lambda\varphi)]}{\rho_1 - \rho_2}, \tag{3.6}$$

where φ is the least in absolute value root of the functional equation $z = \widehat{B}_1(\lambda - \lambda z)$, and

$$\lim_{L\to\infty} p_2(L) = \frac{\rho_2(\rho_1 - 1)}{\rho_1 - \rho_2}. \tag{3.7}$$

Proof. The proof of this theorem follows by application of Lemma 3.1. Straightforward application of the aforementioned lemma to the recurrence relation for $\mathbf{E}\nu_L^{(1)}$ yields the following.

If $\rho_1 < 1$, then

$$\lim_{L\to\infty} \mathbf{E}\nu_L^{(1)} = \frac{1}{1-\rho_1}. \tag{3.8}$$

If $\rho_1 = 1$, then

$$\lim_{L\to\infty} \frac{\mathbf{E}\nu_L^{(1)}}{L} = \frac{2}{\rho_{1,2}}. \tag{3.9}$$

If $\rho_1 > 1$, then

$$\lim_{L\to\infty} \left[\mathbf{E}\nu_L^{(1)} - \frac{1}{\varphi^L(1+\lambda\widehat{B}_1'(\lambda-\lambda z))}\right] = \frac{1}{1-\rho_1}. \tag{3.10}$$

Substituting (3.8)-(3.10) for the limits in (2.8) and (2.9) correspondingly to the cases in the formulation of the theorem finishes the proof. □

4. Further asymptotic analysis of p_1 and p_2

Let us first discuss the statements of Theorem 3.2. Under the assumption $\rho_1 < 1$ we have (3.2) and (3.3). The probability p_1 is positive in limit while the probability p_2 vanishes. Under the assumption $\rho_1 > 1$ we have (3.6) and (3.7). According to these relations the probability p_1 vanishes while p_2 is positive in limit. This means that if both J_1 and J_2 are large positive values proportional to L, then the functional J will take the value proportional to a large parameter L too. Specifically, in the case $\rho_1 < 1$ for this value we have $J \approx (1-\rho_1)J_1$, and in the case $\rho_1 > 1$ we have $J \approx \frac{\rho_2(\rho_1-1)}{\rho_1-\rho_2}J_2$.

In the case $\rho_1 = 1$ both p_1 and p_2 vanish with the rate L^{-1}, and therefore J converges to the limit as $L \to \infty$. Thus, the case $\rho_1 = 1$ is a possible solution of the control problem, while the cases $\rho_1 < 1$ and $\rho_1 > 1$ are irrelevant. Specifically, for $J = J(L)$ we obtain the following:

$$\lim_{L\to\infty} J(L) = j_1\frac{\rho_{1,2}}{2} + j_2\frac{\rho_2}{1-\rho_2}\cdot\frac{\rho_{1,2}}{2}. \tag{4.1}$$

In order to find now the optimal solution consider the following two cases: (i) $\rho_1 = 1+\delta$ and (ii) $\rho_1 = 1-\delta$, where in these both cases $\delta \to 0$ as $L \to \infty$.

In case (i) we have the following two theorems.

Theorem 4.1. *Assume that $\rho_1 = 1+\delta$, $\delta > 0$, and $L\delta \to C > 0$, as $\delta \to 0$ and $L \to \infty$. Assume that $\rho_{1,3} = \rho_{1,3}(\delta)$ is a bounded function of parameter*

δ, for all $0 \leq \delta < 1$ and there exists $\widetilde{\rho}_{1,2} = \lim_{\delta \to 0} \rho_{1,2}(\delta)$. Then,

$$p_1 = \frac{\delta}{e^{2C/\widetilde{\rho}_{1,2}} - 1} + o(\delta), \tag{4.2}$$

$$p_2 = \frac{\delta \rho_2 e^{2C/\widetilde{\rho}_{1,2}}}{(1-\rho_2)(e^{2C/\widetilde{\rho}_{1,2}} - 1)} + o(\delta). \tag{4.3}$$

Proof. The proof of this theorem is similar to that of Theorem 3.4 of [5] and Theorem 4.4 of [6]. Under the conditions of the theorem the following expansion was shown in Subhankulov [18], p. 326:

$$\varphi = 1 - \frac{2\delta}{\widetilde{\rho}_{1,2}} + O(\delta^2). \tag{4.4}$$

Then, by virtue of (4.4) after some algebra we have:

$$1 + \lambda \widehat{B}'(\lambda - \lambda\varphi) = \delta + O(\delta^2). \tag{4.5}$$

Substituting (4.4) and (4.5) for (3.10) we obtain:

$$\mathbf{E}\nu_L^{(1)} = \frac{\mathrm{e}^{2C/\widetilde{\rho}_{1,2}} - 1}{\delta} + O(1). \tag{4.6}$$

From (4.6) and (2.8) and (2.9) we finally obtain the statement of the theorem. □

Theorem 4.2. *Under the conditions of Theorem 4.1 assume that $C = 0$. Then,*

$$\lim_{L\to\infty} Lp_1(L) = \frac{\rho_{1,2}}{2}, \tag{4.7}$$

$$\lim_{L\to\infty} Lp_2(L) = \frac{\rho_2}{1-\rho_2} \cdot \frac{\rho_{1,2}}{2}. \tag{4.8}$$

Proof. The statement of the theorem follows by expanding the main terms of asymptotic relations of (4.2) and (4.3) for small C. □

Notice, that (4.7) and (4.8) coincide with (3.4) and (3.5) correspondingly.

In case (ii) we have the following.

Theorem 4.3. *Assume that $\rho_1 = 1 - \delta$, $\delta > 0$, and $L\delta \to C > 0$, as $\delta \to 0$ and $L \to \infty$. Assume that $\rho_{1,3} = \rho_{1,3}(\delta)$ is a bounded function of parameter δ, for all $0 \leq \delta < 1$ and there exists $\widetilde{\rho}_{1,2} = \lim_{\delta \to 0} \rho_{1,2}(\delta)$. Then,*

$$p_1 = \delta e^{\widetilde{\rho}_{1,2}/2C} + o(\delta), \tag{4.9}$$

$$p_2 = \delta \cdot \frac{\rho_2}{1-\rho_2}\left(e^{\widetilde{\rho}_{1,2}/2C} - 1\right) + o(\delta). \tag{4.10}$$

Proof. From (3.1) we have

$$\sum_{n=0}^{\infty} \mathbf{E}\nu_n^{(1)} z^n = \frac{\widehat{B}_1(\lambda - \lambda z)}{\widehat{B}_1(\lambda - \lambda z) - z}.$$

The sequence $\{\mathbf{E}\nu_n^{(1)}\}$ is an increasing sequence, and in the case $\rho_1 = 1$ from the Tauberian theorem of Hardy-Littlewood (e.g. [15], [16], [18], [19], [20]) we obtain:

$$\lim_{L\to\infty}\frac{\mathbf{E}\nu_L^{(1)}}{L} = \lim_{z\uparrow 1}(1-z)^2\frac{\widehat{B}_1(\lambda-\lambda z)}{\widehat{B}_1(\lambda-\lambda z)-z}.$$

(It is not difficult to check that then (3.9) follows.) Then in the case where $\rho_1 = 1-\delta$, and $L\delta \to C$ as $L\to\infty$, according to the same Tauberian theorem of Hardy and Littlewood, asymptotic behaviour of $\mathbf{E}\nu_L^{(1)}$ can be found from the asymptotic expansion

$$(1-z)\cdot\frac{\widehat{B}_1\left(\lambda-\lambda z\right)}{\widehat{B}_1\left(\lambda-\lambda z\right)-z}, \tag{4.11}$$

as $z\uparrow 1$.

By the Taylor expansion of the denominator of (4.11) we obtain:

$$\begin{aligned}\frac{1-z}{\widehat{B}_1\left(\lambda-\lambda z\right)-z} &\asymp \frac{1-z}{1-z-\rho_1(1-z)+\dfrac{\widetilde{\rho}_{1,2}}{2}(1-z)^2+O\left((1-z)^3\right)}\\ &\asymp \frac{1}{\delta+\dfrac{\widetilde{\rho}_{1,2}}{2}(1-z)+O\left[(1-z)^2\right]}\\ &\asymp \frac{1}{\delta\left(1+\dfrac{\widetilde{\rho}_{1,2}}{2\delta}(1-z)+O\left[(1-z)^2\right]\right)}\\ &\asymp \frac{1}{\delta\exp\left(\dfrac{\widetilde{\rho}_{1,2}}{2\delta}(1-z)\right)}\cdot[1+o(1)].\end{aligned} \tag{4.12}$$

Therefore, assuming that $z=\frac{L-1}{L}\to 1$ as $L\to\infty$, from (4.12) we obtain the asymptotic behaviour of $\mathbf{E}\nu_L^{(1)}$ as $L\to\infty$. We have:

$$\mathbf{E}\nu_L^{(1)} = \frac{1}{\delta e^{\widetilde{\rho}_{1,2}/2C}}\cdot[1+o(1)]. \tag{4.13}$$

Now, substituting (4.13) for (2.8) and (2.9) we obtain the desired statements of the theorem. $\square$

Theorem 4.4. *Under the conditions of Theorem 4.3 assume that $C = 0$. Then we obtain* (4.7) *and* (4.8).

Proof. The statement of the theorem follows by expanding the main terms of asymptotic relations of (4.9) and (4.10) for small C. $\square$

5. **Solution of the control problem**

In this section we formulate the theorem characterizing the solution of control problem.

For $J = J(L)$ we have the following limiting relation

$$\begin{aligned} \lim_{L\to\infty} J(L) &= \lim_{L\to\infty} [p_1(L)J_1(L) + p_2(L)J_2(L)] \\ &= j_1 \lim_{L\to\infty} Lp_1(L) + j_2 \lim_{L\to\infty} Lp_2(L). \end{aligned} \tag{5.1}$$

Substituting (4.1) and (4.2) for the right-hand side of (5.1) and taking into account that $L\delta \to C$, we obtain:

$$\begin{aligned} J^{upper} &= \lim_{L\to\infty} J(L) \\ &= \left[j_1 \frac{1}{e^{2C/\widetilde{\rho}_{1,2}} - 1} + j_2 \frac{\rho_2 e^{2C/\widetilde{\rho}_{1,2}}}{(1-\rho_2)(e^{2C/\widetilde{\rho}_{1,2}} - 1)} \right] \lim_{L\to\infty} L\delta \\ &= C \left[j_1 \frac{1}{e^{2C/\widetilde{\rho}_{1,2}} - 1} + j_2 \frac{\rho_2 e^{2C/\widetilde{\rho}_{1,2}}}{(1-\rho_2)(e^{2C/\widetilde{\rho}_{1,2}} - 1)} \right]. \end{aligned} \tag{5.2}$$

Substituting (4.9) and (4.10) for the right-hand side of (5.1) and taking into account that $L\delta \to C$, we in turn obtain:

$$J^{lower} = C \left[j_1 e^{\widetilde{\rho}_{1,2}/2C} + j_2 \frac{\rho_2}{1-\rho_2} \left(e^{\widetilde{\rho}_{1,2}/2C} - 1 \right) \right]. \tag{5.3}$$

Let us now study the functionals J^{upper} and J^{lower} given by (5.2) and (5.3). Observing (5.2), notice that there contain the constants j_1, j_2 and ρ_2 in (5.2). Let us assume that these constants are given such that

$$j_1 = j_2 \cdot \frac{\rho_2}{1-\rho_2}. \tag{5.4}$$

Then $C = 0$ is the point of min of the functional J^{upper}. Indeed, in this case

$$\begin{aligned} J^{upper} &= j_1 C \left[\frac{1}{e^{2C/\widetilde{\rho}_{1,2}} - 1} + \frac{e^{2C/\widetilde{\rho}_{1,2}}}{e^{2C/\widetilde{\rho}_{1,2}} - 1} \right] \\ &= j_1 C \left[\frac{1}{e^{2C/\widetilde{\rho}_{1,2}} - 1} + \frac{\left(e^{2C/\widetilde{\rho}_{1,2}} - 1\right) + 1}{e^{2C/\widetilde{\rho}_{1,2}} - 1} \right] \\ &= j_1 C \left[1 + \frac{2}{e^{2C/\widetilde{\rho}_{1,2}} - 1} \right]. \end{aligned} \tag{5.5}$$

Therefore in point $C = 0$ we have $\lim_{C\to 0} J^{upper} = j_1 \widetilde{\rho}_{1,2}$, and in the right side of the point $C = 0$ the function J^{upper} is increasing in C. Hence (5.4) is the condition for $C = 0$.

Next,

$$\left[\frac{C}{e^{2C/\widetilde{\rho}_{1,2}} - 1} \right]'_C = \frac{e^{2C/\widetilde{\rho}_{1,2}} - 1 - \dfrac{2C^2}{\widetilde{\rho}_{1,2}} \cdot e^{2C/\widetilde{\rho}_{1,2}}}{(e^{2C/\widetilde{\rho}_{1,2}} - 1)^2}, \tag{5.6}$$

and

$$\left[\frac{C\mathrm{e}^{2C/\widetilde{\rho}_{1,2}}}{\mathrm{e}^{2C/\widetilde{\rho}_{1,2}}-1}\right]'_C = \left[\frac{C}{\mathrm{e}^{2C/\widetilde{\rho}_{1,2}}-1}\right]'_C \mathrm{e}^{2C/\widetilde{\rho}_{1,2}} + \frac{2C}{\widetilde{\rho}_{1,2}}\left[\frac{C\mathrm{e}^{2C/\widetilde{\rho}_{1,2}}}{\mathrm{e}^{2C/\widetilde{\rho}_{1,2}}-1}\right]\mathrm{e}^{2C/\widetilde{\rho}_{1,2}}. \tag{5.7}$$

Clearly that (5.7) is not smaller that (5.6), and they are equal when $C = 0$.

Therefore, if the right-hand side of (5.4) is greater than that left-hand side of (5.4), then $C = 0$ remains to be the value minimizing the functional J^{upper}. The similar result holds for functional J^{lower} given in (5.3). Specifically, if the right-hand side of (5.4) is not greater than the left-hand side of (5.4), then $C = 0$ remains to be the value minimizing the functional J^{lower}.

Thus, the solution of control problem is given by the following theorem.

Theorem 5.1. *If the parameters λ and ρ_2 are given, then the optimal solution of the control problem is the following.*

- *If*

$$j_1 = \frac{\rho_2}{1-\rho_2}j_2,$$

then the optimal solution of the control problem is achieved for $\rho_1 = 1$.

- *If*

$$j_1 > \frac{\rho_2}{1-\rho_2}j_2,$$

then the optimal solution of the control problem is a minimization of the functional J^{upper}. The optimal solution is achieved for $\rho_1 = 1+\delta$, $\delta(L)$ is a small positive parameter, and $L\delta(L) \to C$. C is the nonnegative parameter minimizing (5.2).

- *If*

$$j_1 < \frac{\rho_2}{1-\rho_2}j_2,$$

then the optimal solution of the control problem is a minimization of the functional J^{lower}. The optimal solution is achieved for $\rho_1 = 1-\delta$, $\delta(L)$ is a small positive parameter, and $L\delta(L) \to C$. C is the nonnegative parameter minimizing (5.3).

6. **Concluding remarks**

In this paper we posed and solved a control problem for a large dam. The main specification of the problem is that the performance criteria takes into account passage the lower and upper bounds. The solution of the control problem is asymptotically independent of the explicit form of probability distribution functions $B_1(x)$ and $B_2(x)$, and under the assumption that the parameters λ and ρ_2 are given, in dependence of a performance criteria the parameter ρ_1 must have one of the forms: $\rho_1 = 1$, $\rho_1 = 1+\delta(L)$, or $\rho_1 = 1-\delta(L)$ where $\delta(L) > 0$, and as $L \to \infty$, $\delta(L)$ vanishes and $L\delta(L) \to C$.

Acknowledgement

The research was supported by Australian Research Council grant No. DP0771338.

References

1. Abdel-Hameed, M.S. (2000). Optimal control of a dam using $P^M_{\lambda,\tau}$ policies and penalty cost when the input process is a compound Poisson process with positive drift. *Journal of Applied Probability*, 37, 406-416.
2. Abdel-Hameed, M.S. and Nakhi, Y. (1990). Optimal control of a finite dam using $P^M_{\lambda,\tau}$ policies and penalty cost: total discounted and long-run average cases. *Journal of Applied Probability*, 27, 888-898.
3. Abramov, V.M. (1991). *Investigation of a Queueing System with Service Depending on a Queue-Length*. Donish, Dushanbe, Tadzhikistan. (Russian.)
4. Abramov, V.M. (1997). On a property of a refusals stream. *Journal of Applied Probability*, 37, 800-805.
5. Abramov, V.M. (2002). Asymptotic analysis of the $GI/M/1/n$ queueing system as n increases to infinity. *Annals of Operations Research*, 112, 35-41.
6. Abramov, V.M. (2004). Asymptotic behavior of the number of lost messages. *SIAM Journal on Applied Mathematics* 64 (3) 746-761.
7. Bae, J., Kim, S. and Lee, E.Y. (2002). A P^M_λ policy for an M/G/1 queueing system. *Applied Mathematical Modelling*, 26, 929-939.
8. Bae, J., Kim, S. and Lee, E.Y. (2003). Average cost under the $P^M_{\lambda,\tau}$ policy in a finite dam with compound Poisson inputs. *Journal of Applied Probability*, 40, 519-526.
9. Boxma, O., Kaspi, H., Kella, O. and Perry, D. (2005). On/off storage systems with state-dependent input, output, and switching rates. *Probability in the Engineering and Informational Sciences*, 19, 1-14.
10. Faddy, M.J. (1974). Optimal control of finite dams: discrete (2-stage) output procedure. *Journal of Applied Probability*, 11, 111-121.
11. Kaspi, H., Kella, O., Perry, D. (1996). Dam processes with state-dependent batch sizes, and intermittent production processes with state-dependent rates. *Queueing Systems*, 24, 37-57.
12. Lam, Y. and Lou, J.H. (1987). Optimal control for a finite dam. *Journal of Applied Probability*, 24, 196-199.
13. Lee, E.Y. and Ahn, S.K. (1998). P^M_τ policy for a dam with input formed by a compound Poisson process. *Journal of Applied Probability*, 35, 482-488.
14. Phatarfod, R.M. (1989). Riverflow and reservoir storage models. *Mathematical and Computer Modelling*, 12, 1057-1077.
15. Postnikov, A.G. (1979). *Tauberian Theory and its Application*. Trudy Mat. Inst. Steklov, (2) 144. (Russian).
16. Postnikov, A.G. (1980). *Tauberian Theory and its Application*. Proc. Steklov Math. Inst., (2) 144. (AMS transl. from Russian.)
17. Ross, S.M. (1983). *Stochastic Processes*, John Wiley, New York.
18. Subhankulov, M.A. (1976). *Tauberian Theorems with Remainder*. Nauka, Moscow. (Russian.)
19. Sznajder, R. and Filar, J.A. (1992). Some comments on a theorem of Hardy and Littlewood. *Journal of Optimization Theory and Applications*, 75, 201-208.
20. Takács, L. (1967). *Combinatorial Methods in the Theory of Stochastic Processes*, John Wiley, New York.
21. Zukerman, D. (1977). Two-stage output procedure of a finite dam. *Journal of Applied Probability*, 14, 421-425.

School of Mathematical Sciences, Monash University, Building 28M, Clayton Campus, Clayton, Victoria 3800
E-mail address: vyacheslav.abramov@sci.monash.edu.au